Corrigé Examen RFIDE
18 - 12 - 2009

Exercice 1

1. par déf. $p(x_i = 1 | \mu_i) = \mu_i$

$$\boxed{p(x_i) = \mu_i^{x_i} (1-\mu_i)^{1-x_i}}$$

2. $p(x | \mu) = p(x_1, \dots x_n | \mu)$

$$= \prod_{i=1}^{n} p(x_i | \mu_i)$$

$$\boxed{p(X | \mu) = \prod_{i=1}^{n} \mu_i^{x_i} (1-\mu_i)^{1-x_i}}$$

3 -

3 - 1

$$\boxed{p(C_k | X) = \frac{p(X | C_k)\, p(C_k)}{p(X)}}$$

3. 2 $\log p(x | C_k)\, p(C_k)$

$$\boxed{F(x) = \sum_{i=1}^{n} x_i \log \mu_i + (1-x_i) \log(1-\mu_i) + \log p(C_k)}$$

Exercice 2

1 M. Vraisemblance

1.1 $L(\mu|D) = \prod_{x \in D} p(x|\mu)$

$= \prod_{x} \mu^{x} (1-\mu)^{1-x}$

$= \mu^{n_1} (1-\mu)^{n_0}$

$\log L(\mu|D) = \sum_{x} x \log \mu + (1-x) \log (1-\mu)$

$\frac{\partial \log L}{\partial \mu} = \sum_{x} \frac{x}{\mu} - \frac{(1-x)}{1-\mu}$

$\frac{\partial \log L}{\partial \mu} = 0 \iff \sum_{x} x(1-\mu) - (1-x)\mu = 0$

$\iff \sum x - \mu N = 0$

$\iff \boxed{\mu_{ML} = \frac{n_1}{N}}$ où n_1 est le # de 1 observé en x.

1.2 $\boxed{p(x' = 1|D) = \frac{n_1}{N}}$

1.3 $\boxed{F_{ML}(x) = \sum_{i=1}^{n} x_i \log \frac{n_{i1}}{N} + (1-x_i) \log\left(1 - \frac{n_{i1}}{N}\right) + \log p(c_k)}$

2 -

2 - 1 $\mu_{MAP} = \operatorname{Argmax}_{\mu} \log p(\mu | D)$

$$\log p(\mu|D) = \log \frac{p(D|\mu)\, p(\mu)}{p(D)}$$

$$\operatorname{argmax}_{\mu} \log p(\mu|D) = \operatorname{Argmax}_{\mu} \log\left(p(D/\mu)\, p(\mu)\right)$$

$$\boxed{= \operatorname{Argmax}_{\mu} \sum_{x \in D} \log p(x|\mu) + \log p(\mu)}$$

2 - 2 $\quad p(\mu | \alpha, \beta) = \frac{1}{B(\alpha,\beta)} \mu^{\alpha - 1} (1-\mu)^{\beta - 1}$

$$\log(\ ) = \alpha - 1 \log \mu + \beta - 1 \log(1-\mu) - \log B$$

$$\frac{dL}{d\mu} + \frac{d \log p(\mu)}{d\mu}$$

$$= \sum_{x} \frac{x}{\mu} - \frac{1-x}{1-\mu} + \frac{\alpha - 1}{\mu} - \frac{\beta - 1}{1-\mu}$$

$$= \frac{n_1}{\mu} - \frac{n_0}{1-\mu} + \frac{\alpha - 1}{\mu} - \frac{\beta - 1}{1-\mu}$$

$$A = \frac{n_1 + \alpha - 1}{\mu} - \frac{n_0 + \beta - 1}{1-\mu}$$

$$A = 0 \iff (1-\mu)(n_1 + \alpha - 1) - \mu(n_0 + \beta - 1) = 0$$

$$\boxed{\mu_{MAP} = \frac{n_1 + \alpha - 1}{n_1 + n_0 + \alpha + \beta - 2}}$$

2-3 — si petit # obs. le prior est déterminant
si non grd —— tend vers μ_{ML}

2.4.1 $p(x'=1|D) = \mu_{MAP}$

2-4.2

$$\boxed{F_{MAP}(X) = \sum_{i=1}^{n} x_i \log \frac{n_1+\alpha-1}{n_1+n_0+\alpha+\beta-2} + (1-x_i)\log \frac{n_0+\beta-1}{n_1+n_0+\alpha+\beta-2}}$$

3 —

3.1

$$p(D|\alpha,\beta) = \int_0^1 p(D|\mu)\,p(\mu|\alpha,\beta)\,d\mu$$

$$= \int_0^1 \mu^{n_1}(1-\mu)^{n_0} \frac{1}{B(\alpha,\beta)} \mu^{\alpha-1}(1-\mu)^{\beta-1} d\mu$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 \mu^{n_1+\alpha-1}(1-\mu)^{n_0+\beta-1} d\mu$$

$$\boxed{P(D|\alpha,\beta) = \frac{B(\alpha+n_1, \beta+n_0)}{B(\alpha,\beta)}}$$

3.2

$$p(x'=1|D,\alpha,\beta) =$$

$$\frac{B(\alpha+n_1+1, \beta+n_0)}{B(\alpha,\beta)} \times \frac{B(\alpha,\beta)}{B(\alpha+n_1,\beta+n_0)}$$

$$= \frac{B(\alpha+n_1+1, \beta+n_0)}{B(\alpha+n_1, \beta+n_0)}$$

$$= \frac{\Gamma(\alpha+n_1+1)\,\cancel{\Gamma(\beta+n_0)}}{\Gamma(\alpha+n_1+1+\beta+n_0)} \times \frac{\Gamma(\alpha+n_1+\beta+n_0)}{\Gamma(\alpha+n_1)\,\cancel{\Gamma(\beta+n_0)}}$$

$$= \frac{(\alpha+n_1)\,\cancel{\Gamma(\alpha+n_1)}}{(\alpha+n_1+\beta+n_0)\,\cancel{\Gamma(\alpha+n_1+\beta+n_0)}} \times \frac{\cancel{\Gamma(\alpha+n_1+\beta+n_0)}}{\cancel{\Gamma(\alpha+n_1)}}$$

$$\boxed{p(x'=1|D,\alpha,\beta) = \frac{\alpha+n_1}{\alpha+n_1+\beta+n_0}}$$

$$P(X) = \prod_{i=1}^{n} p(x_i)$$

$$= \prod_{i} p(x_i=1|D,\alpha,\beta)^{x_i} \times p(x_i=0|D,\alpha,\beta)^{1-x_i}$$

$$= \prod_{i=1}^{n} \left(\frac{n_{i1}+\alpha}{n_{i1}+n_{i0}+\alpha+\beta}\right)^{x_i} \left(\frac{n_{i0}+\beta}{n_{i1}+n_{i0}+\alpha+\beta}\right)^{1-x_i}$$

Fonct. disc.

$$\boxed{F_{\text{Bayes}}(X) = \sum_{i=1}^{n} x_i \log\left(\frac{n_{i1}+\alpha}{n_{i1}+n_{i0}+\alpha+\beta}\right) + (1-x_i)\log\left(\frac{n_{i0}+\beta}{n_{i1}+n_{i0}+\alpha+\beta}\right)}$$