1. $p(x | p) = \prod_{i=1}^{D} p(x_i | p_i)$

$= \prod_{i=1}^{D} p_i^{x_i} (1-p_i)^{(1-x_i)}$

2. $p(x, p, \pi) = \sum_{k=1}^{K} \pi_k \, p(x | p_k)$

2.1 x image, p_k vecteur de paramètres de Bernoulli de la $k^{ième}$ composante du mélange, π_k proba. a priori de la $k^{ième}$ composante

2-2 1)• choisir une composante suivant π_k
2)• générer l'image suivant p_k

2.3 $p(X | p, \pi) = \prod_{n=1}^{N} \sum_{k=1}^{K} \pi_k \, p(x_n | p_k)$

$\log p(X | p, \pi) = \sum_{n=1}^{N} \log \left(\sum_{k=1}^{K} \pi_k \, p(x_n | p_k) \right)$

2-4

cf cours, eq. couplées.

~~4~~ 2

3.1 $p(x|z,p) = \prod_{k=1}^{K} p(x|p_k)^{z_k}$

$p(z|\pi) = \prod_{k=1}^{K} \pi_k^{z_k}$

3.2

$$p(x,z|p,\pi) = p(x|z,p,\pi)\, p(z|\pi)$$

$$\sum_z p(x,z|p\,\pi) = \sum_z p(x|z,p,\pi)\, p(z|\pi)$$
$$= \sum_z \prod_{k=1}^{K} p(x|p_k)^{z_k} \pi_k^{z_k}$$
$$= \sum_{k=1}^{K} p(x|p_k)\, \pi_k$$

3.3

$$\log(p(X,Z|p,\pi)) = \log \prod_{n=1}^{N} p(x_n,z_n|p,\pi)$$
$$= \sum_{n=1}^{N} \log p(x_n,z_n|p,\pi)$$
$$= \sum_{n=1}^{N} \log \prod_{k=1}^{K} (p(x_n|p_k)\pi_k)^{z_{nk}}$$
$$= \sum_{n=1}^{N} \sum_{k=1}^{K} z_{nk}(\log \pi_k + \log p(x_n|p_k))$$
$$= \sum_{n=1}^{N} \sum_{k=1}^{K} z_{nk}\left(\log \pi_k + \sum_{i=1}^{D} x_{ni} \log p_{ki} + (1-x_{ni})\log(1-p_{ki})\right)$$

3

3.4

$$E_{Z|X,p,\pi}\left[\log p(X,Z|p,\pi)\right] =$$

$$\sum_{n=1}^{N}\sum_{k=1}^{K} E_{Z|X,p,\pi}[z_{nk}]\left(\log \pi_k + \sum_{i=1}^{D} x_{ni}\log p_{ki} + (1-x_{ni})\log(1-p_{ki})\right)$$

$$E_{Z|X,p,\pi}[z_{nk}] = \sum_{Z} z_{nk}\, p(Z|X,\dots)$$

$$= \sum_{Z-z_{nk}} \Bigg(\underbrace{\sum_{z_{nk}=0}^{1} z_{nk}\, p(z_{nk}|x_n,p,\pi)}_{p(z_{nk}=1|x_n,\cdot)}\Bigg) p(Z-z_{nk}|\dots)$$

$$\text{et } \sum_{Z-z_{nk}} p(Z-z_{nk}|\dots) = 1$$

$$\boxed{E_{Z|X,p,\pi}[z_{nk}] = p(z_{nk}=1|x_n,\dots) = \boxed{\gamma_{nk}}}$$

$$\boxed{E_{Z|X,p,\pi}\left[\log p(X,Z|p,\pi)\right] = \sum_{n=1}^{N}\sum_{k=1}^{K}\gamma_{nk}\left(\log \pi_k + \sum_{i=1}^{D} x_{ni}\log p_{ki} + (1-x_{ni})\log(1-p_{ki})\right)}$$

3.5

- $$\gamma_{nk} = p(z_{nk}=1|x_n) = \frac{p(x_n|z_{nk}=1)\,p(z_{nk}=1)}{p(x_n)}$$

$$\boxed{\gamma_{nk} = \frac{\pi_k\, p(x_n|p_k)}{\sum_{j=1}^{K}\pi_j\, p(x_n|p_j)}}$$

4. ~~$\#$~~ 4

- posons $N_k = \sum_{n=1}^{N} \gamma_{nk}$: # pts associé à la composante k.

$$\bar{x}_k = \frac{1}{N_k} \sum_{n=1}^{N} \gamma_{nk} x_n$$

4.1. Lagrangien associé à $\max Q$ sc $\sum_{k=1}^{K} \pi_k = 1$

$$\mathcal{L}(Q) = Q + \lambda \left(\sum_{k=1} \pi_k - 1 \right)$$

4.2.
$$\frac{\partial \mathcal{L}(Q)}{\partial p_{ki}} = \sum_{n=1}^{N} \gamma_{nk} \left(\frac{x_{ni}}{p_{ki}} - \frac{(1-x_{ni})}{1-p_{ki}} \right]$$

$$= \frac{1}{p_{ki}(1-p_{ki})} \sum_{n=1}^{N} \gamma_{nk} (x_{ni} - x_{ni} p_{ki} - p_{ki} + x_{ni} p_{ki})$$

$$= \frac{1}{p_{ki}(1-p_{ki})} \sum_{n=1}^{N} \gamma_{nk} (x_{ni} - p_{ki})$$

$$\frac{\partial \mathcal{L}(Q)}{\partial p_{ki}} = 0 \iff \sum_{n=1}^{N} \gamma_{nk} x_{ni} = p_{ki} \sum_{n=1}^{N} \gamma_{nk}$$

$$\iff \boxed{p_{ki} = \bar{x}_{ni}}$$

$$\implies \boxed{p_k = \bar{x}_k}$$

4.3 $$\frac{\partial L}{\partial \pi_k} = \sum_{n=1}^{N} \gamma_{nk} \frac{1}{\pi_k} + \lambda$$

$$\frac{\partial L}{\partial \pi_k} = 0 \iff \sum_{n=1}^{N} \gamma_{nk} + \lambda \pi_k = 0$$

en sommant sur tous les k $$\sum_{k=1}^{K} \sum_{n=1}^{N} \gamma_{nk} + \lambda \underbrace{\sum_{k=1}^{K} \pi_k}_{1} = 0$$

$$\boxed{\begin{aligned} \lambda &= \sum_{k=1}^{N} N_k = N \\ \text{et } \pi_k &= \frac{1}{\lambda} \sum_{n=1}^{N} \gamma_{nk} = \frac{N_k}{N} \end{aligned}}$$

5. EM cf cours